HIGHER-ORDER ALGEBRAIC THEORIES AND RELATIVE MONADS

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(Masaryk University Algebra Seminar, 13.05.21)

Outline

- Algebraic theories
- Second-order algebraic theories
- Higher-order algebraic theories
- A universal characterisation of Lawn
- Relative monads, monads, and theories
- 0^{th}- order algebraic theories

I. ALGEBRAIC THEORIES

First-order operators

1. $$\frac{\Gamma \vdash a \qquad \Gamma \vdash b}{\Gamma \vdash a \times b}$$ Multiplication

2. $$\frac{\Gamma \vdash a}{\Gamma \vdash a^{-1}}$$ Inverses

3. $$\frac{\Gamma \vdash m : M \qquad \Gamma \vdash a : A}{\Gamma \vdash m * a : A}$$ Actions

Algebraic theories

identity-on-objects cartesian functor

$$\begin{tikzcd} \mathbb{L} \arrow[r, "L"] & \mathcal{L} \end{tikzcd}$$

free cartesian category on a point

cartesian category

(Here, 'cartesian' means finite products.)

Algebraic theories

$$\begin{tikzcd} \mathbb{L} \arrow[r, "L"] & \mathcal{L} \end{tikzcd}$$

The objects of $\mathcal{L}$ are given by X^n for X the generating object, and $n \in \mathbb{N}$.

A morphism $X^n \xrightarrow{\vec{t}} X^m$ represents an m-tuple of terms in n variables:

$$\langle x_1, \dots, x_n \vdash t_i \rangle_{1 \leq i \leq m}$$

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$$\langle x_1, \dots, x_n \vdash t_i \rangle_{1 \leq i \leq m}$$

n-ary operation $\langle t_i : X^n \longrightarrow X \rangle_i$

Algebraic theories

A map of algebraic theories is a commutative triangle

$$\begin{tikzcd}
\mathcal{L} \arrow[rr, "F"] & & \mathcal{L}' \\
& \mathbb{L} \arrow[ul, "L"] \arrow[ur, "L'"'] &
\end{tikzcd}$$

Algebraic theories and their maps form a category Law.

Law

Law is a well-behaved category.

- It has all small limits and colimits.
- More than that, it is locally strongly finitely presentable:

Law

Law is a well-behaved category.

- It has all small limits and colimits.
- More than that, it is locally strongly finitely presentable:

$$\mathrm{Law} \simeq \mathrm{Sind}(\mathcal{C}) \simeq \mathrm{Cart}(\mathcal{C}^{\circ}, \mathrm{Set})$$

for some cocartesian $\mathcal{C}$

Sind = cocompletion under sifted colimits

Monads and theories

There is a classic equivalence between algebraic theories and (strongly) finitary monads on the category of sets.

$$\mathrm{Law} \simeq \mathrm{Mnd}_f(\mathrm{Set}) = \mathrm{Mnd}_{sf}(\mathrm{Set})$$

Finitary = preserves filtered colimits

Strongly finitary = preserves sifted colimits (sifted-cocontinuous)

Universal algebra

$$\begin{tikzcd}
& \text{Equational logic} \arrow[dl, bend left] \arrow[dr, bend left] \arrow[dl, phantom, "\simeq"] \arrow[dr, phantom, "\simeq"] & \\
\text{Algebraic theories} \arrow[ur, bend left] \arrow[rr, bend left] \arrow[rr, phantom, "\simeq"] & & \text{Monads on Set} \arrow[ul, bend left] \arrow[ll, bend left]
\end{tikzcd}$$

II. SECOND-ORDER ALGEBRAIC THEORIES

[Fiore & Mahmoud, 2010]

Second-order operators

1. $$\frac{\Gamma, x \vdash f \qquad \Gamma \vdash x_0}{\Gamma \vdash \frac{df}{dx}(x_0)}$$

Differential operators (cf. Plotkin 2020)

2. $$\frac{\Gamma, x \vdash P}{\Gamma \vdash \exists x . P}$$

Logical quantifiers

3. $$\frac{\Gamma, x \vdash t}{\Gamma \vdash \lambda x . t}$$

λ-abstraction

Second-order operators

4. $$\frac{\Gamma \vdash t : A + B \quad \Gamma, a : A \vdash u : C \quad \Gamma, b : B \vdash v : C}{\Gamma \vdash \mathrm{case}(t, a.u, b.v) : C}$$

Coproducts, case-splitting

5. $$\frac{\Gamma, x : X \vdash f : X}{\Gamma \vdash \mathrm{fix}(f) : X}$$

Fixed points

6. Parameterised algebraic theories [Staton, 2013]

operators
and equations
of the theory

$$\begin{tikzcd}\mathbb{L} \arrow[r, "L"] & \mathcal{L}\end{tikzcd}$$

structural
operations

Second-order theory of equality

$\mathbb{L}_2$ is the free cartesian category with an exponentiable object (i.e. an object such that $(-)^X : \mathbb{L}_2 \to \mathbb{L}_2$ exists).

Objects of $\mathbb{L}_2$ are given by products

$$X^{X^{n_1}} \times \cdots \times X^{X^{n_k}}$$

with morphisms given by projection and evaluation.

Second-order theory of equality

$\mathbb{L}_2$ is the free cartesian category with an exponentiable object (i.e. an object such that $(-)^X : \mathbb{L}_2 \to \mathbb{L}_2$ exists).

Objects of $\mathbb{L}_2$ are given by products

$$X^{\overbrace{X^{n_1}}} \times \dots \times X^{\overbrace{X^{n_k}}}$$

exponents are the objects of $\mathbb{L}$

with morphisms given by projection and evaluation.

Second-order algebraic theories

identity-on-objects cartesian functor, preserving the exponentiable object

$$\begin{tikzcd}\mathbb{L}_2 \arrow[r, "L"] & \mathcal{L}\end{tikzcd}$$

free cartesian category on an exponentiable object

cartesian category with an exponentiable object

Second-order algebraic theories

$$\begin{tikzcd}\mathbb{L}_2 \arrow[r, "L"] & \mathcal{L}\end{tikzcd}$$

A morphism $X^{X^{n_1}} \times \cdots \times X^{X^{n_k}} \xrightarrow{\vec{t}} X^{X^{m_1}} \times \cdots \times X^{X^{m_l}}$ in $\mathcal{L}$ represents an l-tuple of terms in k metavariables and m_i variables:

$$\left\langle (x^1_1, \ldots, x^1_{n_1})\, x_1 \,,\ \ldots \,,\ (x^k_1, \ldots, x^k_{n_k})\, x_k \,,\ y_1, \ldots, y_{m_i} \vdash t \right\rangle_i$$

parameterised variable

Ordinary variable

‘Differentiate $f(x)$ with respect to x and evaluate at x_0.’

$$\partial(x.\ f(x), x_0)$$

represented by

$$X^X \times X \xrightarrow{\partial} X$$

Second-order algebraic theories

A map of second-order algebraic theories is a commutative triangle

$$\begin{tikzcd} \mathcal{L} \arrow[rr, "F"] & & \mathcal{L}' \\ & \mathbb{L}_2 \arrow[ul, "L"] \arrow[ur, "L'"'] & \end{tikzcd}$$

Second-order algebraic theories and their maps form a category $\mathbf{Law}_2$.

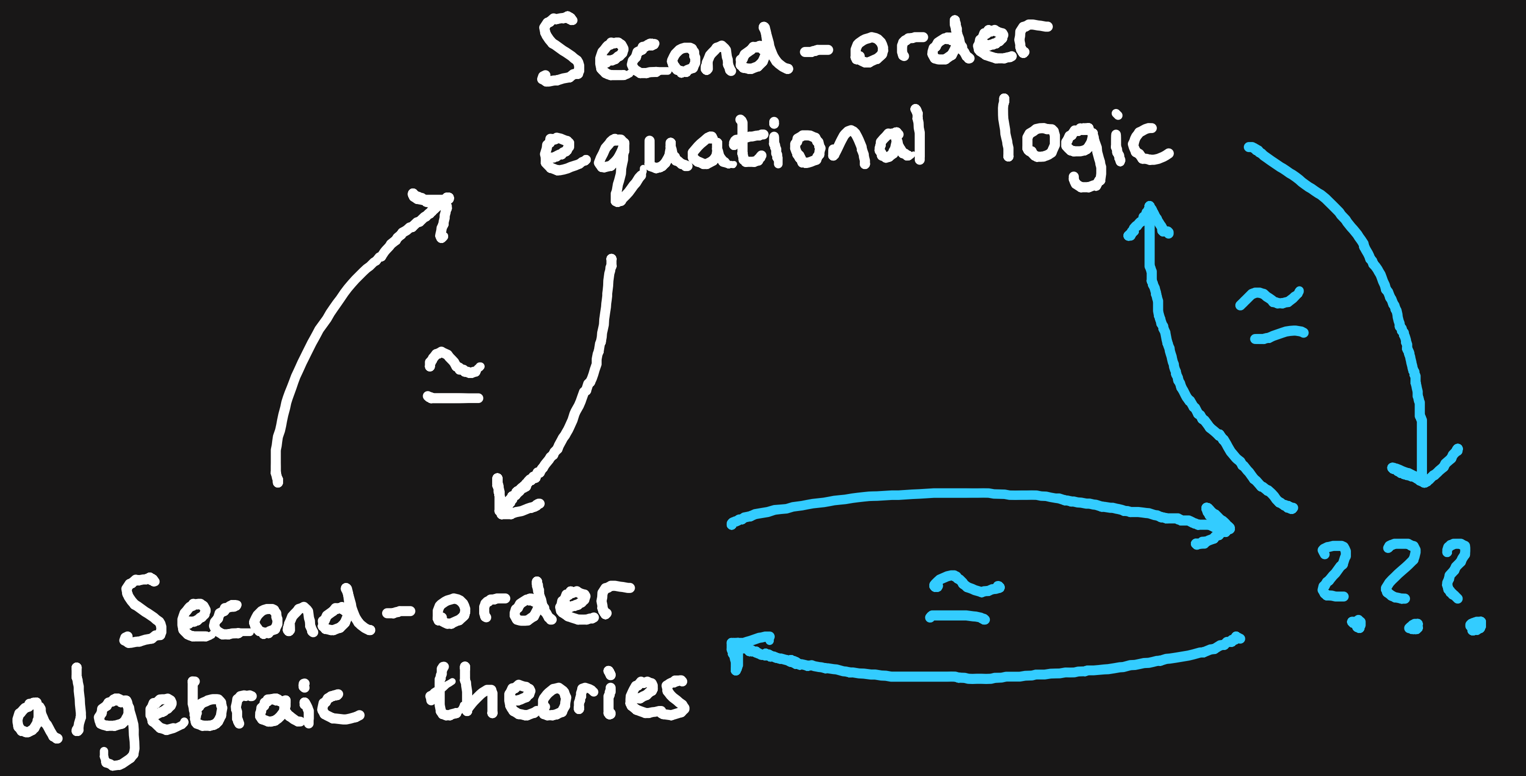
Second-order algebraic theories

How well-behaved is Law_2?

Does it have:

- Limits?
- Colimits?
- A universal property?

Second-order universal algebra



Second-order equational logic

Second-order algebraic theories

???

≃

≃

≃

III. HIGHER-ORDER ALGEBRAIC THEORIES

Third-order operators

1. Continuations
2. Selection operators

operators
and equations
of the theory

$$\mathbb{L} \xrightarrow{L} \mathcal{L}$$

structural
operations

Higher-order theory of equality

$\mathbb{L}_n$ is the free cartesian category on an **n-tetrable object** (i.e. an object X such that

$$\underbrace{\overset{1}{1}, \overset{2}{X}, \overset{3}{X^X}, \overset{4}{X^{X^X}}, \overset{\ldots}{\ldots}}_{n}$$

is exponentiable).

Higher-order theory of equality

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$$\underbrace{\underset{1}{1}, \underset{2}{X}, \underset{3}{X^X}, \underset{4}{X^{X^X}}, \underset{\ldots}{\ldots}}_{n}$$

is exponentiable).

Intuitively, morphisms in $\mathbb{L}_n$ represent operators taking operators as operands.

Higher-order theory of equality

$\mathbb{L}_n$ is the free cartesian category on an n-tetrable object (i.e. an object X such that $1, X, X^X, X^{X^X}, \ldots$ is exponentiable).

We have:

$$\mathbb{L} = \mathbb{L}_1 \hookrightarrow \mathbb{L}_2 \hookrightarrow \cdots \hookrightarrow \mathbb{L}_\omega$$

$\uparrow$ free cartesian category on a point

$\uparrow$ free cartesian-closed category on a point

Higher-order algebraic theories

identity-on-objects cartesian functor, preserving the n-tetrable object

$$\begin{tikzcd} \mathbb{L}_n \arrow[r, "L"] & \mathcal{L} \end{tikzcd}$$

free cartesian category on an n-tetrable object

cartesian category with an n-tetrable object

Higher-order algebraic theories

A map of n^{th}-order algebraic theories is a commutative triangle

$$\begin{tikzcd}
\mathcal{L} \arrow[rr, "F"] & & \mathcal{L}' \\
& \mathbb{L}_n \arrow[ul, "L"] \arrow[ur, "L'"'] &
\end{tikzcd}$$

n^{th}-order algebraic theories and their maps form a category Law_n.

1. How well-behaved is Lawn?
2. Is there a monad correspondence?

Exponentiable subcategories

A (full) subcategory $\mathscr{C}' \overset{j}{\hookrightarrow} \mathscr{C}$ is **exponentiable** if, for all $X \in \mathscr{C}'$, jX is exponentiable in $\mathscr{C}$,

i.e. $jX \times (-) \dashv (-)^{jX}$.

Exponentiable subcategories

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Ex

$\amalg_n \hookrightarrow \amalg_{n+1}$ is exponentiable.

Exponentiable subcategories

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Denote by **Exp_n** the category with

$$\begin{tikzcd}
\mathscr{C}_1 \arrow[r, hook] \arrow[d, "F_1"'] & \cdots \arrow[r, hook] & \mathscr{C}_n \arrow[d, "F_n"] \\
\mathscr{D}_1 \arrow[r, hook] & \cdots \arrow[r, hook] & \mathscr{D}_n
\end{tikzcd}$$

where F_{i+1} preserves exponentiation by objects of $\mathscr{C}_i$

Prop.

There is an adjunction

$$\begin{tikzcd} \mathrm{Exp}_n \arrow[r, hook, shift left=1.5, "{\triangleright \_ \triangleleft}"] \arrow[r, phantom, "\perp"] & \mathrm{Exp}_{n+1} \arrow[l, shift left=1.5, "{\vartriangle \_ \vartriangle}"] \end{tikzcd}$$

where

Conservative cartesian closure of $\mathcal{C}_n$

$$^{\triangleright}\mathcal{C}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{C}_n{}^{\triangleleft} = \mathcal{C}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{C}_n \overset{}{\dashrightarrow} \tilde{\mathcal{C}}$$

$$_{\vartriangle}\mathcal{C}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{C}_{n+1}{}_{\vartriangle} = \mathcal{C}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{C}_n$$

Prop.

Let $\mathcal{C}_1 \overset{j}{\hookrightarrow} \mathcal{C}_2$ be an object of Exp_2.

Strict cartesian functors $\tilde{\mathcal{C}}_2 \longrightarrow \mathrm{Set}$ are in bijection with strict cartesian identity-on-objects functors out of $\mathcal{C}_2$ preserving exponentiation by objects of $\mathcal{C}_1$.

$$\mathrm{Cart}(\tilde{\mathcal{C}}_2, \mathrm{Set}) \cong j /_{io} \mathrm{Exp}_2$$

Prop.

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Strict cartesian functors $\widetilde{\mathcal{C}}_2 \longrightarrow \mathrm{Set}$ are in bijection with strict cartesian identity-on-objects functors out of $\mathcal{C}_2$ preserving exponentiation by objects of $\mathcal{C}_1$.

$$\mathrm{Cart}(\widetilde{\mathcal{C}}_2, \mathrm{Set}) \cong j /_{\mathrm{io}} \mathrm{Exp}_2$$

Sketch

For $F : \widetilde{\mathcal{C}}_2 \longrightarrow \mathrm{Set}$, $F(Y^X)$ defines a hom-set, with composition induced by evaluation.

Cor

$\mathbb{L}_n \overset{j}{\hookrightarrow} \mathbb{L}_{n+1}$ is an object of Exp_2, so strict cartesian functors $\widetilde{\mathbb{L}_{n+1}} \longrightarrow \mathrm{Set}$ are in bijection with strict cartesian identity-on-objects functors out of $\mathbb{L}_{n+1}$ preserving exponentiation by objects of $\mathbb{L}_n$.

$$\begin{tikzcd}
\mathrm{Cart}(\widetilde{\mathbb{L}_{n+1}}, \mathrm{Set}) \cong & j /_{io} \mathrm{Exp}_2 = \mathrm{Law}_{n+1} & & \\
\mathbb{L}_{n+2} \arrow[u, phantom, "\wr"] & \mathbb{L}_{n+1} \arrow[r] \arrow[u, bend left] & \mathcal{L}' & (A) \\
 & \mathbb{L}_n \arrow[u, "j"] \arrow[r] & \mathcal{L} \arrow[u] & (B)
\end{tikzcd}$$

The universal property of Law_n

Thm

Law_n is locally strongly finitely presentable.

$$\mathrm{Law}_n \simeq \mathrm{Cart}(\mathbb{L}_{n+1}, \mathrm{Set})$$

$$\simeq \mathrm{Sind}(\mathbb{L}_{n+1}^{\circ})$$

sifted cocompletion

free cartesian category on an (n+1)-tetrable point

The universal property of Law_n $(n=1)$

Thm

Law_1 is locally strongly finitely presentable.

$$\begin{aligned} \mathrm{Law} = \mathrm{Law}_1 &\simeq \mathrm{Cart}(\mathbb{L}_2 \, , \mathrm{Set}) \\ &\simeq \mathrm{Sind}(\mathbb{L}_2^{\circ}) \end{aligned}$$

sifted cocompletion

free cartesian category on an exponentiable object

The universal property of Law_n

Thm

Law_n is locally strongly finitely presentable.

$$
\begin{aligned}
\mathrm{Law}_n &\simeq \mathrm{Cart}(\mathbb{L}_{n+1}, \mathrm{Set}) \\
&\simeq \mathrm{Sind}(\mathbb{L}_{n+1}^{\circ})
\end{aligned}
$$

sifted cocompletion

free cartesian category on an (n+1)-tetrable point

Hence also:

- Locally finitely presentable
- Cocomplete
- Complete

The universal property of Law_n

Thm

Law_n is locally strongly finitely presentable.

$$\begin{aligned} \mathrm{Law}_n &\simeq \mathrm{Cart}(\mathbb{L}_{n+1}, \mathrm{Set}) \\ &\simeq \mathrm{Sind}(\mathbb{L}_{n+1}^{\circ}) \end{aligned}$$

sifted cocompletion

free cartesian category on an $(n+1)$-tetrable point

Hence also:

- Locally finitely presentable
- Cocomplete
- Complete

(Cf. Uemura 2020, 'The universal exponentiable arrow'.)

Coreflections

Thm

There is a coreflection of categories:

inclusion of presentations

$$\begin{tikzcd}
\mathrm{Law}_n \arrow[r, hook, bend left, "\ulcorner - \urcorner"] \arrow[r, phantom, "\perp"] & \mathrm{Law}_{n+1} \arrow[l, bend left, "\llcorner - \lrcorner"]
\end{tikzcd}$$

discard $(n+1)^{th}$- order terms

Coreflections

Thm

There is a coreflection of categories

$$\begin{tikzcd}
\mathrm{Sind}(\mathbb{L}_{n+1}) \simeq \mathrm{Law}_n \arrow[r, hook, bend left, "\ulcorner - \urcorner"] \arrow[r, phantom, "\perp"] & \mathrm{Law}_{n+1} \simeq \mathrm{Sind}(\mathbb{L}_{n+2}) \arrow[l, bend left, "\llcorner - \lrcorner"]
\end{tikzcd}$$

$\mathbb{L}_{n+1} \xhookrightarrow{j} \mathbb{L}_{n+2}$ induces the algebraic functor $\llcorner - \lrcorner$, which has a left adjoint by abstract nonsense.

Coreflections

There is a chain of coreflections,

$$\begin{tikzcd}
\mathrm{Law}_1 \arrow[r, hook, shift left=1.5] \arrow[r, phantom, "\perp"] & \mathrm{Law}_2 \arrow[l, shift left=1.5] \arrow[r, hook, shift left=1.5] \arrow[r, phantom, "\perp"] & \cdots \arrow[l, shift left=1.5] \arrow[r, hook, shift left=1.5] \arrow[r, phantom, "\perp"] & \mathrm{Law}_\omega \arrow[l, shift left=1.5]
\end{tikzcd}$$

allowing us to freely extend or restrict the order of a higher-order algebraic theory.

IV. RELATIVE MONADS

Relative adjunctions

$$\begin{tikzcd}
 & \mathcal{D} \arrow[dr, "R"] & \\
\mathcal{C}' \arrow[ur, "L"] \arrow[rr, "J"'] & \dashv & \mathcal{C}
\end{tikzcd}$$

$L \mathbin{{}_J\!\dashv} R$ when $\mathcal{D}(Lx, y) \cong \mathcal{C}(Jx, Ry)$ natural in $x \in \mathcal{C}'$ and $y \in \mathcal{D}$.

Relative monads

A J-relative monad $(T, \eta, (-)^*)$ consists of

- a function $T : |\mathcal{C}'| \longrightarrow |\mathcal{C}|$
- a transformation $\eta_X : JX \longrightarrow TX$
- a transformation $(-)^*_{X,Y} : \mathcal{C}(JX, TY) \longrightarrow \mathcal{C}(TX, TY)$

satisfying unitality and associativity conditions.

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Prop: a monad is precisely an Id-relative monad.

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Prop: a monad is precisely an Id-relative monad.

Prop: every relative adjunction induces a relative monad.

Simple slice category

Let $L : \mathbb{L}_n \to \mathcal{L}$ be an nth-order algebraic theory.

Define a functor

$$L /\!/ (-) \ : \ \mathcal{L}^{\circ} \longrightarrow \mathrm{Law}_n$$

where $\mathcal{L} /\!/ X$ is the simple slice category over X:

$$\mathcal{L} /\!/ X (A, B) = \mathcal{L}(A \times X, B)$$

Simple slice category

Let $L : \mathbb{L}_n \longrightarrow \mathcal{L}$ be an n^{th}-order algebraic theory.

Define a functor

$$L /\!/ (-) \; : \; \mathcal{L}^{\circ} \longrightarrow \mathrm{Law}_n$$

where $\mathcal{L}/\!/X$ is the simple slice category over X:

$$\mathcal{L}/\!/X(A, B) = \mathcal{L}(A \times X, B)$$

which is the free cartesian category with a morphism $1 \longrightarrow X$ containing $\mathcal{L}$.

Theories to relative monads

Lem

Let $L : \mathbb{L}_{n+1} \longrightarrow \mathcal{L}$ be an $(n+1)^{th}$-order algebraic theory.
There is a よ-relative adjunction:

$$\begin{tikzcd}
& \mathcal{L}^\circ \arrow[dr, "\lfloor \mathcal{L} /\!/ - \rfloor"] \arrow[d, phantom, "\dashv"] & \\
\mathbb{L}_{n+1}^\circ \arrow[ur, "L^\circ"] \arrow[rr, hook, "よ"'] & {} & \mathrm{Sind}(\mathbb{L}_{n+1}^\circ)
\end{tikzcd}$$

Hence L induces a よ-relative monad.

Relative monads to theories

Lem

Let $T : \mathbb{L}_{n+1}^{\circ} \longrightarrow \mathrm{Sind}(\mathbb{L}_{n+1}^{\circ})$ be a よ-relative monad. There is a よ-relative adjunction:

$$\begin{tikzcd}
& \mathrm{Kl}(T) \arrow[dr, "U"] & \\
\mathbb{L}_{n+1}^{\circ} \arrow[ur, "K"] \arrow[rr, "\text{よ}"'] & \dashv & \mathrm{Sind}(\mathbb{L}_{n+1}^{\circ})
\end{tikzcd}$$

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\end{tikzcd}$$

Kleisli inclusion preserves coproducts

K° is identity-on-objects and cartesian...

Relative monads to theories

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\end{tikzcd}$$

K° is identity-on-objects and cartesian...
but may not preserve exponentials :(

Interlude

When does the Kleisli inclusion preserve coexponentials?

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When does the Kleisli inclusion preserve coexponentials?

$$
\begin{aligned}
& \mathrm{Kl}(T)(X, Y+Z) \\
= \ & \mathrm{Law}_n(X, T(Y+Z)) \\
\overset{?}{\cong} \ & \mathrm{Law}_n(X, Y+TZ) \\
\cong \ & \mathrm{Law}_n(X_Y, TZ) \\
= \ & \mathrm{Kl}(T)(X_Y, Z)
\end{aligned}
$$

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\cong \; & \mathrm{Law}_n(X_Y, TZ) \\
= \; & \mathrm{Kl}(T)(X_Y, Z)
\end{aligned}
$$

Hence we require

$$T(Y+Z) \cong Y+TZ \quad (Y, Z \in \mathbb{L}_{n-1})$$

Call such relative monads

+-linear.

Interlude

When does the Kleisli inclusion preserve coexponentials?

$$
\begin{aligned}
& \mathrm{Kl}(T)(X, Y+Z) \\
= \; & \mathrm{Law}_n(X, T(Y+Z)) \\
\cong^{?} \; & \mathrm{Law}_n(X, Y+TZ) \\
\cong \; & \mathrm{Law}_n(X_Y, TZ) \\
= \; & \mathrm{Kl}(T)(X_Y, Z)
\end{aligned}
$$

Hence we require

$$T(Y+Z) \cong Y+TZ \quad (Y, Z \in \mathbb{L}_{n-1})$$

Call such relative monads **+-linear**.

(+-linearity is trivial for $n=1$.)

Relative monads to theories

Lem

Let $T : \mathbb{L}_{n+1}^{\circ} \longrightarrow \mathrm{Sind}(\mathbb{L}_{n+1}^{\circ})$ be a よ-relative monad. There is a よ-relative adjunction:

$$\begin{tikzcd}
 & \mathrm{Kl}(T) \arrow[dr, "U"] & \\
\mathbb{L}_{n+1}^{\circ} \arrow[ur, "K"] \arrow[rr, hook, "よ"'] & \dashv & \mathrm{Sind}(\mathbb{L}_{n+1}^{\circ})
\end{tikzcd}$$

Relative monads to theories

Lem

Let $T : \mathbb{L}_{n+1}^{\circ} \longrightarrow \mathrm{Sind}(\mathbb{L}_{n+1}^{\circ})$ be a +-linear よ-relative monad. There is a よ-relative adjunction:

$$\begin{tikzcd}
& \mathrm{Kl}(T) \arrow[dr, "U"] & \\
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\end{tikzcd}$$

Relative monads to theories

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Let $T : \mathbb{L}_{n+1}^{\circ} \longrightarrow \mathrm{Sind}(\mathbb{L}_{n+1}^{\circ})$ be a $+$-linear よ-relative monad. There is a よ-relative adjunction:

$$\begin{tikzcd}
& \mathrm{Kl}(T) \arrow[dr, "U"] \arrow[d, phantom, "\dashv"] & \\
\mathbb{L}_{n+1}^{\circ} \arrow[ur, "K"] \arrow[rr, hook, "\text{よ}"'] & {} & \mathrm{Sind}(\mathbb{L}_{n+1}^{\circ})
\end{tikzcd}$$

$K^{\circ} : \mathbb{L}_{n+1}^{\circ} \longrightarrow \mathrm{Kl}(T)^{\circ}$ is an $(n+1)^{\text{th}}$-order algebraic theory.

Thm

$$\mathrm{Law}_{n+1} \simeq \mathrm{RMnd}_{+\text{-lin}}\left(\text{よ}_{\amalg_{n+1}^{\circ}}\right)$$

$(n+1)^{\text{th}}$-order algebraic theories

$+$-linear $(\amalg_{n+1}^{\circ} \hookrightarrow \mathrm{Law}_n)$-relative monads

Thm

$$\mathrm{Law}_{n+1} \simeq \mathrm{RMnd}_{+\text{-lin}}\left(よ_{\amalg_{n+1}{}^{\circ}}\right)$$

$(n+1)^{th}$-order algebraic theories

+-linear ($\amalg_{n+1}{}^{\circ} \hookrightarrow \mathrm{Law}_n$)-relative monads

(But what about ordinary monads?)

Relative monads & monads

Thm

Let $\mathcal{C}$ be a locally strongly finitely presentable category. There is an equivalence of categories

$$\mathrm{RMnd}(\mathcal{C}_{sfp} \hookrightarrow \mathcal{C}) \simeq \mathrm{Mnd}_{sf}(\mathcal{C})$$

Thm

$$\mathrm{Law}_{n+1} \simeq \mathrm{RMnd}_{+\text{-lin}}\left(よ_{\mathbb{L}_{n+1}^{\circ}}\right)$$

$(n+1)^{th}$-order algebraic theories

+-linear $(\mathbb{L}_{n+1}^{\circ} \hookrightarrow \mathrm{Law}_n)$-relative monads

$$\simeq \mathrm{Mnd}_{+\text{-lin},\, sf}(\mathrm{Law}_n)$$

sifted-cocontinuous +-linear monads on Law_n

Prop.

Let $L : \mathbb{L}_{n+1} \to \mathcal{L}$ be an $(n+1)^{th}$-order algebraic theory.

The corresponding monad is given by

$$T_L(X) \qquad = \llcorner L + \ulcorner X \urcorner \lrcorner$$

Prop.

Let $L : \mathbb{L}_{n+1} \to \mathcal{L}$ be an $(n+1)^{\text{th}}$-order algebraic theory. The corresponding monad is given by

$$T_L(X) = \llcorner L + \ulcorner X \urcorner \lrcorner$$

When $n=0$, this says that T_L takes a set of constants, freely adds them to L, then extracts the new constants formed from those in X under the operations of L.

Prop.

Let $L : \mathbb{L}_{n+1} \longrightarrow \mathcal{L}$ be an $(n+1)^{\text{th}}$-order algebraic theory. The corresponding monad is given by

$$T_L(X) = \llcorner L + \ulcorner X \urcorner \lrcorner$$

$$T_L(X)(B,C) \cong \int^{A \in \mathbb{L}_{n+1}} \mathcal{L}(A, C^B) \times \ulcorner X \urcorner (1, A)$$

Algebras

Let $L : \mathbb{L}_{n+1} \to \mathcal{L}$ be an $(n+1)^{th}$-order algebraic theory, and let $T_L : \mathrm{Law}_n \longrightarrow \mathrm{Law}_n$ be the corresponding monad.

$$T_L\text{-}\mathrm{Alg} \simeq \mathrm{Cart}(\mathcal{L}, \mathrm{Set})$$

0^{th}-order algebraic theories

It is well-known that (first-order) algebraic theories correspond to (strongly) finitary monads on Set.

Since $\mathbf{Law}_{n+1} \simeq \mathbf{Mnd}_{+\text{-lin},\text{sf}}(\mathbf{Law}_n)$, and $+$-linearity is trivial when $n < 2$, we are led to conclude that

$$\mathbf{Law}_0 \simeq \mathbf{Set}$$

How may we interpret this syntactically?

0^{th}-order algebraic theories

A 0^{th}-order algebraic theory is a strict, terminal object-preserving identity-on-objects functor

$$\{1 \leftarrow X\} \longrightarrow \mathbb{L}_0 \xrightarrow{L} \mathcal{L}$$

for which every morphism in $\mathcal{L}$ is constant (i.e. factors through 1).

0^{th}-order algebraic theories

A 0^{th}-order algebraic theory is a strict, terminal object-preserving identity-on-objects functor

$$\begin{tikzcd} \{1 \leftarrow X\} \arrow[r] & \mathbb{L}_0 \arrow[r, "L"] & \mathcal{L} \end{tikzcd}$$

for which every morphism in $\mathcal{L}$ is constant (i.e. factors through 1).

Equivalently, a T-operad for T the terminal monad on Set.

0^{th}-order algebraic theories

0^{th}-order algebraic theories are theories of constants. Each $L : \mathbb{L}_0 \longrightarrow \mathcal{L}$ defines a set $\mathcal{L}(1, X)$, and vice versa, exhibiting an isomorphism of categories

$$\mathrm{Law}_0 \cong \mathrm{Set}$$

0^{th}-order algebraic theories

Recall that:

$$\mathrm{Law}_n \simeq \mathrm{Sind}(\mathbb{L}_{n+1}^{\circ}) \qquad (n > 0)$$

Substituting $n = 0$:

$$\begin{aligned}\mathrm{Law}_0 &\simeq \mathrm{Sind}(\mathbb{L}_1^{\circ}) \\ &\simeq \mathrm{Sind}(\mathrm{FinSet}) \\ &\simeq \mathrm{Set}\end{aligned}$$

Everything previously discussed remains valid for $n = 0$.

Cor.

$$\mathrm{Law}_n \simeq \mathrm{Mnd}_{+\text{-lin},\,sf}^{\;n}(\mathrm{Law}_0)$$

Summary

- Higher-order algebraic theories generalise algebraic theories by (higher-order) variable binding operators.
- There are coreflections $$\begin{tikzcd} \mathrm{Law}_n \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & \mathrm{Law}_{n+1} \arrow[l, shift left] \end{tikzcd}$$.
- $\mathrm{Law}_n \simeq \mathrm{Sind}(\mathbb{L}_{n+1}^{\circ})$
- $\mathrm{Law}_{n+1} \simeq \mathrm{Mnd}_{sf,+\text{-}lin}(\mathrm{Law}_n)$
- Sets are 0th-order algebraic theories.